Geometric multigrid with plane smoothing for thin elements in 3-D magnetostatic field calculation

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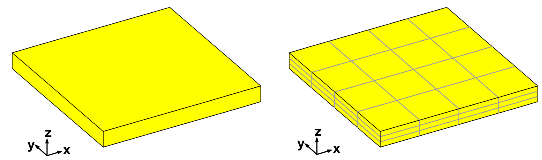
***Abstract*—The geometric multigrid method (MGM) using the point-wise iterative method as the smoother converges slowly for solving 3-D magnetostatic problems discretized by the finite element method (FEM) with thin elements in the mesh. This paper proposes an edge based plane smoother which can significantly improve the convergence of the MGM. Numerical examples show that the MGM with the proposed smoother retains its good efficiency even for edge elements with extremely high aspect ratios.**



(a) Macro element mesh. (b) Finite element mesh.

Fig. 1. A magnetic thin plate meshed by (a) macro elements and (b) finite elements.

I. Introduction

In the numerical analysis of 3-D magnetostatic problems discretized by the FEM, iterative methods such as the incomplete Cholesky conjugate gradient (ICCG) method and the geometric MGM converge slowly when thin elements are used in the FEM mesh. The convergence characteristics of ICCG have been improved by adding unknown variables to near parallel edges and using the singularity decomposition technique (SDT) [1]. For the MGM which has an advantage over the ICCG for problems with a large number of equations, its efficiency for slightly bad quality meshes has been improved by using the symmetric Gauss-Seidel preconditioned conjugate gradient (SGSCG) smoother in the multigrid algorithm [2], but it deteriorates for extremely thin elements. The reason for this is that the smoothing effect of the point-wise smoother such as Gauss-Seidel is poor with respect to the anisotropic direction in the mesh.

It has been found that for problems discretized by the FEM with anisotropic meshes, the plane smoother has a better convergence than the point-wise smoother for the Poisson equation solved by the MGM for nodal elements [3]. This paper proposes an edge based plane smoother where the edge elements in the same anisotropic direction in the FEM mesh are grouped in a block and their corresponding unknowns are updated simultaneously in one smoothing iteration. The convergence characteristics of the proposed smoother in the multigrid algorithm will be investigated by an academic problem: a thin magnetic plate with varying thickness surrounded by a coil. As a practical application, the box shield model introduced in [4] is analyzed. Numerical results show that substantial improvement of the solution time can be obtained by using the MGM with the plane smoother as the preconditioner of the conjugate gradient method.

II. Finite element formulation

The differential equation for the static magnetic field can be formulated as

$$\mathtt{curl}(\nu\mathtt{curl}\boldsymbol{A}) = \mathtt{curl}\boldsymbol{T}_0 \tag{1}$$

where ν is the magnetic reluctivity, $\boldsymbol{A}$ is the magnetic vector potential defined by $\boldsymbol{B} = \mathtt{curl}\boldsymbol{A}$, and the divergence free source current density is described by the curl of the impressed vector potential $\boldsymbol{T}_0$.

Approximating the vector potential by n_e edge basis functions $\boldsymbol{N}_i$ and applying Galerkin techniques to (1) result in the following algebraic equations,

$$\int_\Omega \mathtt{curl}\boldsymbol{N}_i \cdot \nu\mathtt{curl}\boldsymbol{A}_h d\Omega = \int_\Omega \mathtt{curl}\boldsymbol{N}_i \cdot \nu\boldsymbol{T}_0 d\Omega \qquad i = 1, 2, ..., n_e \tag{2}$$

where $\boldsymbol{A}_h = \sum_{i=1}^{n_e} a_i\boldsymbol{N}_i$ is the approximation of the vector potential. Eq. (2) can be written in a matrix form $A\boldsymbol{x} = \boldsymbol{b}$.

III. Multigrid Smoother

In the geometric MGM, a hierarchy of FEM meshes has to be constructed. In our case, the geometric structure is described by a brick shaped so-called macro element that constitutes the coarse grid of the MGM as shown in Fig. 1(a). The macro element is automatically subdivided in all three directions to obtain hexahedral finite elements regarded as the fine mesh of the MGM as shown in Fig. 1(b). On each level of finite element meshes, a cheap iterative method called smoother is implemented for elimination of the high frequency errors. For the point-wise smoother such as Gauss-Seidel or its acceleration by the conjugate gradient method, the smoothing effect is poor with respect to the anisotropic direction, e.g. the x direction in the anisotropic mesh in Fig. 2 which depicts the mesh in one of x-z planes of the macro element in Fig. 1(b).

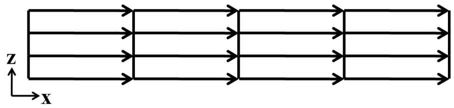


Fig. 2. The mesh of one of the x-z planes in the macro element.

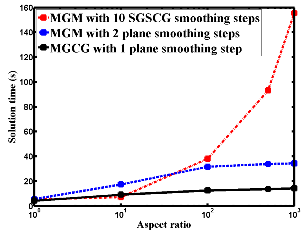


Fig. 3. Solution times of different methods versus the aspect ratio.

This difficulty can be overcome by including all edges in x direction in the mesh shown in Fig. 2 in one block and updating the corresponding unknowns simultaneously by the following block Gauss-Seidel iteration,

$$\begin{aligned} &\boldsymbol{x}^{(v)} = \boldsymbol{x}^{(v-1)} - A_v^{-1}(A\boldsymbol{x}^{(v-1)} - \boldsymbol{b}), \\ &v = 1, 2, ..., n_b \end{aligned} \tag{3}$$

where v is the order number of the block, n_b is the number of blocks and A_v^{-1} is the inverse matrix of the submatrix A_v of A corresponding to the block.

For problems solved by the MGM, the procedure introduced above has to be applied to each macro element where the edges in the anisotropic direction of the mesh in each plane discretized by anisotropic grids are included in one block. The construction of the blocks and the corresponding submatrices, as well as the calculation of the inverse of the submatrices need to be done on each level of FEM meshes during preprocessing.

IV. Numerical Examples

A. Magnetic thin plate

The convergence of the MGM with the plane smoother for the thin plate in Fig. 1(b) will be investigated. Second order hexahedral finite elements with 36 edges have been used for the calculation. The relative permeability of the plate is $\mu_r = 1 \times 10^3$ and the imposed magnetic field in z direction is produced by a current-fed coil around the plate. The aspect ratio in the mesh of the plate can be adjusted from 1 to 1000 by simply varying the thickness of the plate.

Fig. 3 compares the solution times of the MGM with different smoothers. It can be seen that two plane smoothing iterations are sufficient for efficient reduction of the errors and the MGM with the plane smoothing has a stable convergence with the increase of the aspect ratio. Moreover, the conjugate gradient method preconditioned by the multigrid (MGCG) with only one plane smoothing iteration results in further improvement of the efficiency as depicted by the solid line in Fig. 3.

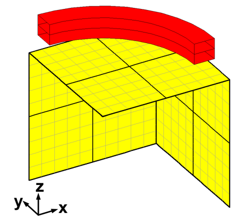


Fig. 4. Geometry and finite element discretization of the box and the coil.

TABLE I
SOLUTION DATA FOR THE BOX SHIELD MODEL

	MGM with SGSCG	MGCG with plane smoothing
Number of equations	254072	254072
Number of blocks	0	9086
Preprocessing time (s)	27.768	60.567
Iterations	566	15
Solution time (s)	2167.627	68.391

B. Box shield model

As a practical example in Fig. 4, the box shield model [4] is analyzed. Only the linear case with a relative permeability of 1000 has been considered. The thickness of the shield plate is 0.01mm and the plate is divided into two layers of finite elements. The maximum aspect ratios in the mesh of the plate and the air region are 2000 and 10000, respectively.

In Table I, the number of blocks and iterations, the corresponding preprocessing and solution times of the MGM with the SGSCG smoother and the MGCG with the plane smoother have been compared. Although the preprocessing time of the plane smoother is higher than that of the SGSCG smoother as the inverse of the submatrices has to be calculated, the solution time has been reduced by more than 95% by the MGCG with the plane smoother.

V. Conclusion

In this paper, we present a plane smoother for edge elements used in the multigrid algorithm for thin elements in magnetostatic problems. Numerical results have shown that the proposed smoother is cheap but efficient for smoothing the errors and independent of the quality of the mesh. In particular the decrease of the solution time by using the multigrid with the plane smoother as the preconditioner of the conjugate gradient has been demonstrated.

References

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